Phys 410
Fall 2013
Lecture #28 Summary
12 December, 2013

We continued our discussion of Special Relativity. Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value c in every direction in all inertial reference frames.

We reviewed the Lorentz transformation for the description of the same event from the perspective of two inertial reference frames S and S' moving at speed V relative to each other:

$$x' = \gamma(x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - xV/c^2)$$

It was noted that this transformation has the appearance of a rotation in a 4-dimensional space spanned by the coordinates x_1, x_2, x_3 (the re-named ordinary Cartesian coordinates) and a new coordinate $x_4 = ct$. The Lorentz transformation can be written in "rotational" form as $x'^{(4)} = \overline{\overline{\Lambda}}\, x^{(4)}$, where $x^{(4)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is the space-time 4-vector [which can also be written as $x^{(4)} = (\vec{x}, ct)$, for example] and the 'rotation' matrix representing the Lorentz transformation is $\overline{\overline{\Lambda}} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$. This is not the most general Lorentz transformation. It is a special case called a "boost", which corresponds to a pair of reference frames moving relative to each other along one of the coordinate axes. Note that we use the superscript $x^{(4)}$ to denote 4-vectors and the vector sign ($\vec{x}$) to denote ordinary 3-vectors.

A four-vector is any quantity that transforms under a Lorentz transformation the same way that the space-time 4-vector transforms. Other 4-vectors that we will encounter include velocity and momentum, to be defined later. The first postulate of relativity implies that the laws of physics have the same form in all inertial reference frames. In other words there is no inertial frame in which the laws of physics are particularly simple (e.g. having fewer terms) than any other frame. This suggests that the laws of physics should be Lorentz invariant, meaning that they take exactly the same form after Lorentz transformation. In other words, the laws of physics should be formulated in terms of 4-vectors! Our objective now is to formulate relativistic mechanics in terms of 4-vectors, and to make sure that they reduce to classical Newtonian form in the limit $\frac{V}{c} \ll 1$.

The length of a three dimensional vector $(x^2 + y^2 + z^2)$ does not change after the coordinate system describing the vector is rotated (i.e. $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$). This is an example of a scalar invariant of the vector (namely $\vec{x} \cdot \vec{x}$). There is also a scalar product of 4-vectors that always has the same value after an arbitrary Lorentz transformation. The scalar length of a 4-vector is defined as $s \equiv x_1^2 + x_2^2 + x_3^2 - x_4^2$. This can also be written as $s = \vec{x} \cdot \vec{x} - (ct)^2$. Note the minus sign in the last term. This is dictated by the form of the Lorentz transformation given above. We showed by direct calculation with the Lorentz transformation that $s' = x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2 = s$, proving that this is the correct definition of a Lorentz-invariant scalar. Note that s is a scalar that can be positive, negative, or zero.

We applied this scalar invariant to describe the spherical wave emanating from an exploding firework. In frame S at rest relative to the shell just before it explodes, the leading edge of the expanding light sphere is given by the equation $r^2 = (ct)^2$, which is nicely described by the invariant $s = r^2 - (ct)^2 = 0$. Now consider a frame S' moving by at a high rate of speed V relative to S. The scalar invariant in this frame has exactly the same value: $s' = s = 0$. In other words it says that $r'^2 - (ct')^2 = 0$, which means that an observer in S' also sees the light expanding in a spherical manner at the same speed as the observer in S! This counter-intuitive result is clearly in accordance with the second postulate of relativity.

We started in to relativistic dynamics with an effort to define relativistic momentum. We expect the laws of physics to have the same form in all reference frames, hence they should be Lorentz invariant. The easiest way to do this is to formulate the laws in terms of 4-vector quantities. They should also reduce to familiar Newtonian forms in the small-velocity limit.

Mass is defined to be an invariant quantity (all inertial reference frames agree on its value) and it is equal to the rest mass. Ordinary 3-momentum is not Lorentz invariant. 3-momentum that is conserved in a collision witnessed in one reference frame will not be conserved in another one moving by at relativistic speed (see problem 15.54). We need to develop a 4-vector version of momentum. Start with $x^{(4)} = (\vec{x}, ct)$, and consider taking a derivative with respect to time. The problem is that different inertial observers cannot agree on the evolution of time, hence we need a version of time that all observers can agree upon. This would be the proper time interval dt_0 which is the differential change in time when the particle of interest is at rest in your reference frame, corresponding to a differential 4-vector of $dx_0^{(4)} = (0, c\,dt_0)$. Comparing the invariant length of this 4-vector to that of a general differential displacement $dx^{(4)} = (\vec{v}, c)dt$ yields $dt_0 = dt/\gamma(v)$, where $\gamma(v) = 1/\sqrt{1-(v/c)^2}$ is the γ-factor associated with particle's velocity as measured in a given reference frame.

With this we can define a *bone-fide* velocity 4-vector that transforms like the space-time 4-vector: $u^{(4)} = \frac{dx^{(4)}}{dt_0} = \gamma(v)(\vec{v}, c)$. We define the associated Lorentz-invariant momentum as $p^{(4)} = m\gamma(v)(\vec{v}, c)$. Note that the 3-vector part of this reduces to the ordinary Newtonian momentum in the $\frac{v}{c} \ll 1$ limit, as required. Note that momentum now carries a fourth component – is this excess baggage or something useful? Recall Noether's theorem (and the idea of ignorable coordinates in Lagrangians), which says that the homogeneity of space implies linear momentum conservation. Likewise the homogeneity of time implies conservation of energy. In this case the time-like component of the momentum 4-vector is defined as relativistic energy E divided by the speed of light, $p^{(4)} = (m\gamma(v)\vec{v}, E/c)$. In other words $E = \gamma(v)mc^2$.

To examine the plausibility of this assignment of the relativistic energy, look at its value in the small-velocity limit $\frac{v}{c} \ll 1$. In this limit $E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \cong mc^2 + \frac{1}{2}mv^2 + \cdots$, through a binomial expansion of the denominator. The first term is called the rest energy and is a constant as far as classical Lagrangian mechanics is concerned, hence it plays no role in Newtonian dynamics. The second term is the Newtonian kinetic energy that we have employed since the get-go. Thus this definition of energy reduces to our familiar one in the low-speed Newtonian limit. The relativistic kinetic energy can be written as $T = E - mc^2 = (\gamma(v) - 1)mc^2$.